Homework 3

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ABSTRACT

This homework has four parts, one theoretical and three computational.

1. Theoretical questions related to Fourier transform and forward interpolation.
2. Data compression using 2-D Fourier transform.
3. Data interpolation after coordinate transformation.
4. Analyzing your own data.

PREREQUISITES

Completing the computational part of this homework assignment requires

- `Madagascar` software environment available from
  `http://www.ahay.org/`
- LaTeX environment with `SEG`TeX available from
  `http://www.ahay.org/wiki/SEGTeX`

To do the assignment on your personal computer, you need to install the required environments. Please ask for help if you don't know where to start.

The homework code is available from the `Madagascar` repository by running

```
svn co http://svn.code.sf.net/p/rsf/code/trunk/book/geo391/hw3
```

THEORETICAL PART

You can either write your answers to theoretical questions on paper or edit them in the file `hw3/paper.tex`. Please show all the mathematical derivations that you perform.

1. Show that, using the helix transform and imposing helical boundary conditions, it is possible to compute a 2-D digital Fourier transform using 1-D FFT program. Assuming that the input data is of size $N \times N$, would this approach have any computational advantages?

2. The Taylor series expansion of the inverse sine function around zero is

$$\arcsin x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \cdots \tag{1}$$

   (a) Show how one can use expansion (1) to design a digital filter that approximates the derivative operator.

   Hint: Use the identity $1/Z - Z = 2\,i\,\sin(\omega\,\Delta t)$.

   (b) In particular, find a seven-point derivative filter of the form

$$D(Z) = d_{-3}/Z^3 + d_{-2}/Z^2 + d_{-1}/Z + d_0 + d_1\,Z + d_2\,Z^2 + d_3\,Z^3\ . \tag{2}$$

3. The parabolic B-spline $\beta_2(x)$ is a function defined as

$$\beta_2(x) = \int\limits_{-\infty}^{\infty} \beta_1(t)\,\beta_0(x-t)\,dt\ , \tag{3}$$

where

$$\beta_0(x) = \begin{cases} 1 & \text{for} \quad |x| \le 1/2 \\ 0 & \text{for} \quad |x| > 1/2 \end{cases} \tag{4}$$

and

$$\beta_1(x) = \int\limits_{-\infty}^{\infty} \beta_0(t)\,\beta_0(x-t)\,dt\ = \begin{cases} 1-|x| & \text{for} \quad |x| \le 1 \\ 0 & \text{for} \quad |x| > 1 \end{cases} \tag{5}$$

   (a) Find an explicit expression for $\beta_2(x)$.

   (b) Show that decomposing a continuous data function $d(x)$ into the convolution basis with parabolic B-spines

$$d(x) = \sum_k c_k\,\beta_2(x-k) \tag{6}$$

   leads to an interpolation filter of the form

$$Z^\sigma \approx B_2(Z) = \frac{a_0(\sigma)\,Z^{-1} + a_1(\sigma) + a_2(\sigma)\,Z}{b_0\,Z^{-1} + b_1 + b_2\,Z}\ . \tag{7}$$

   Define $a_0(\sigma)$, $a_1(\sigma)$, $a_2(\sigma)$, b_0, b_1, and b_2.

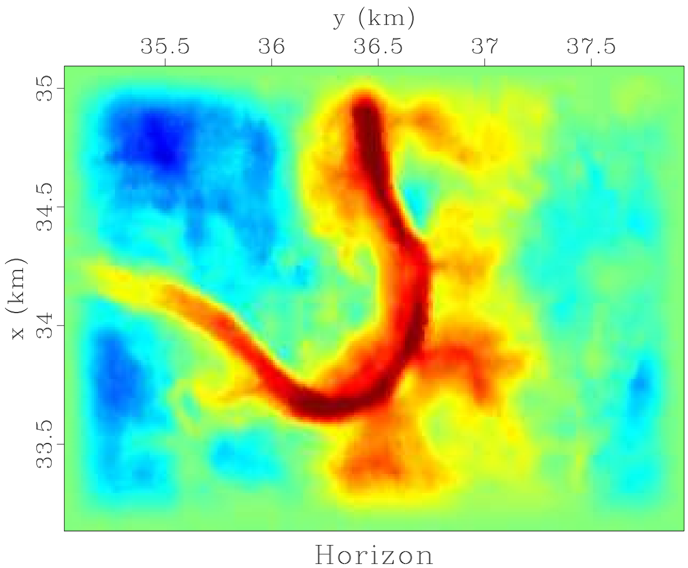


Figure 1: Seismic depth slice with a channel structure.

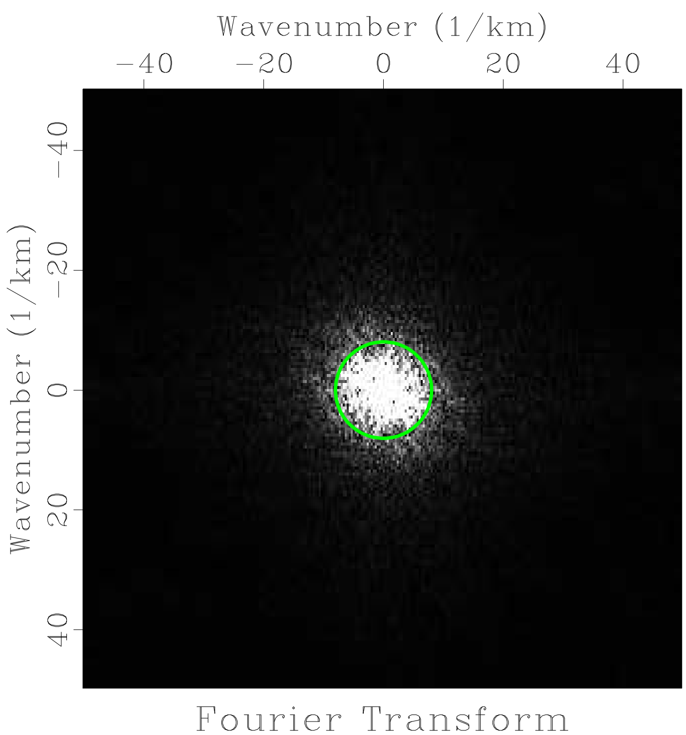


Figure 2: Absolute value of the Fourier transform of the seismic slice from Figure 1. The circle inside shows a window selected for compression.

FOURIER COMPRESSION

In this exercise, we will use a depth slice selected from a 3-D seismic volume and shown in Figure 1 (Hall, 2007). Notice a channel structure.

The goal of your assignment is to find a compressed representation of the data in the Fourier transform domain. Figure 2 shows the Fourier transform of the data from Figure 1. We can see that most of the energy gets concentrated near the center (zero frequency).

There are two alternative ways to compress data in the Fourier domain:

- One approach is to select a range of frequencies that contain the most important information. An advantage of this approach is the ability to subsample the original data by transforming back from a windowed range of frequencies. The results from this method are shown in Figure 3.
- Another approach is to zero all Fourier coefficients below a certain threshold value, regardless of which frequencies they represent. The results from this method are shown in Figure 4. Figure 5 shows the selected threshold plotted against the histogram of Fourier coefficients.

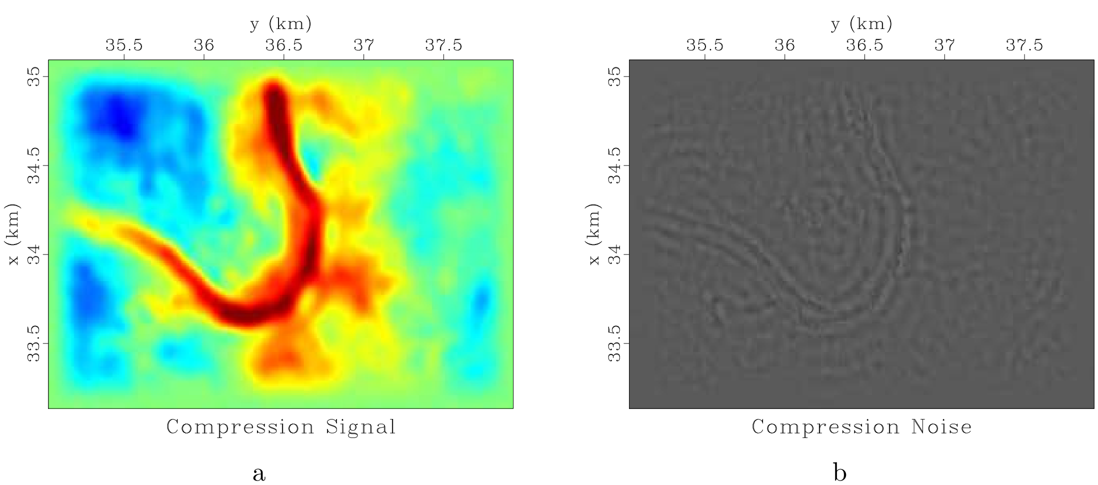


Figure 3: Data separated into signal (a) and noise (b) by applying Fourier compression with windowing.

1. Change directory to `hw3/fourier`.
2. Run

   ```
   scons view
   ```

   to reproduce the figures on your screen.

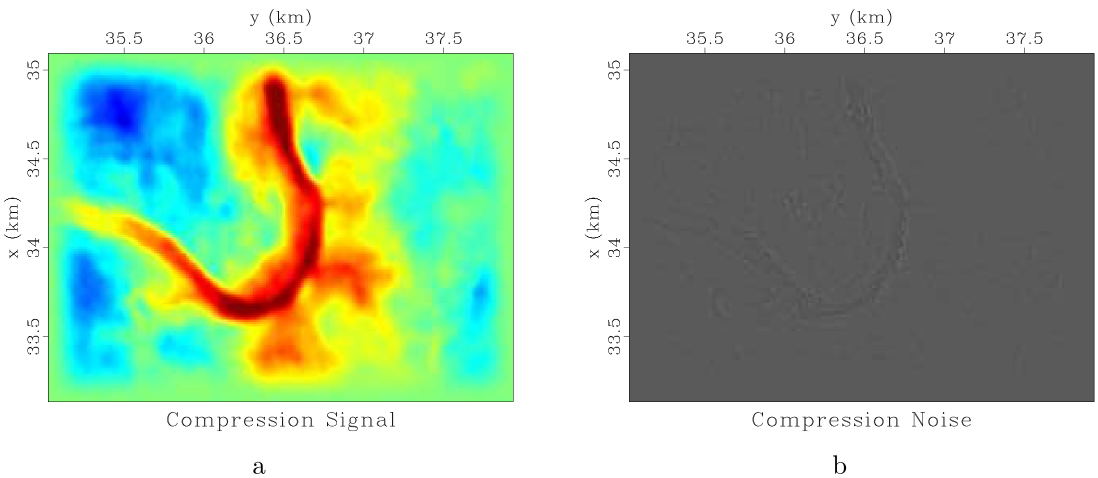


Figure 4: Data separated into signal (a) and noise (b) by applying Fourier compression with thresholding.

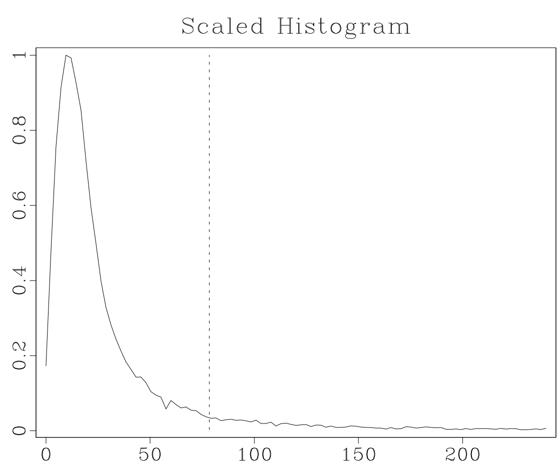


Figure 5: Normalized histogram of Fourier coefficients (by absolute value). The vertical line shows the selected threshold.

3. Modify the `SConstruct` file to decrease the size of the window so that the noise level increases in Figure 3b. How do you measure the noise level? Find a level that you find negligibly small.

4. Modify the `SConstruct` file to increase the threshold value so that the compression achieves the same quality as in the previous case. The noise level in Figure 4b should match that in Figure 3b.

5. Compare the number of nonzero Fourier coefficients in both cases. Which method achieves a better compression?

6. **EXTRA CREDIT** for finding a way for a better compression of the data in the Fourier domain. Your data reconstruction should have the same noise level, yet the number of non-zero coefficients in the Fourier domain should be smaller.

```
from rsf.proj import *

# Get data
##########
Fetch('horizon.asc','hall')

# Convert format
Flow('data','horizon.asc',
     '''
     echo in=$SOURCE data_format=ascii_float n1=3 n2=57036 |
     dd form=native | window n1=1 f1=-1 | add add=-65 |
     put
     n2=291 o2=35.031 d2=0.01 label2=y unit2=km
     n1=196 o1=33.139 d1=0.01 label1=x unit1=km |
     costaper nw1=25 nw2=25
     ''')

# Display
def plot(title):
    return '''
    grey color=j title="%s"
    transp=y yreverse=n clip=14
    ''' % title
Result('data',plot('Horizon'))

# 2-D Fourier Transform
########################
Flow('fft','data',
     'rtoc | fft3 axis=1 pad=1 | fft3 axis=2 pad=1')
Plot('fft',
     '''
```

```
      math output="abs(input)" | real |
      grey title="Fourier Transform" allpos=y screenratio=1
      ''')

# A. Compression by Windowing
################################

cut = 8 # !!! CHANGE ME !!!

# Create a frame
Flow('frame','fft','real | math output="sqrt(x1*x1+x2*x2)" ')
Plot('frame',
      '''
      contour nc=1 c0=%g plotfat=5 plotcol=3
      wantaxis=n wanttitle=n screenratio=1
      ''' % cut)
Result('fft','fft frame','Overlay')

# Cut a hole
Flow('fcut','frame fft',
      '''
      mask max=%g |
      dd type=float | rtoc |
      mul ${SOURCES[1]}
      ''' % cut)

# Inverse FFT
Flow('sig','fcut',
      'fft3 axis=2 inv=y | fft3 axis=1 inv=y | real')
Result('sig',plot('Compression Signal'))

Flow('cut','data sig','add scale=1,-1 ${SOURCES[1]}')
Result('cut',plot('Compression Noise') + 'color=I')

# B. Compression by Thresholding
###################################

thr = 80 # !!! CHANGE ME !!!

# Plot histogram
Plot('hist','fft',
      '''
      math output="abs(input)" | real |
      histogram o1=0 d1=%g n1=101 |
      dd type=float | scale axis=1 |
```

```
    graph title="Scaled Histogram" pad1=n
    label1= unit1= label2= unit2=
    ''' % (0.03*thr))
Flow('line.asc',None,
    'echo 0 0 0 1 n1=4 data_format=ascii_float in=$TARGET')
Plot('line','line.asc',
    '''
    dd type=complex form=native |
    graph min1=-1 max1=2 plotcol=5
    wantaxis=n wanttitle=n dash=1
    ''')
Result('hist','hist line','Overlay')

# Thresholding
Flow('fthr','fft','thr thr=%g' % thr)

# Inverse FFT
Flow('thr','fthr',
    'fft3 axis=2 inv=y | fft3 axis=1 inv=y | real')
Result('thr',plot('Compression Signal'))

# Subtract from Data
Flow('noi','data thr','add scale=1,-1 ${SOURCES[1]}')
Result('noi',plot('Compression Noise') + 'color=I')

End()
```

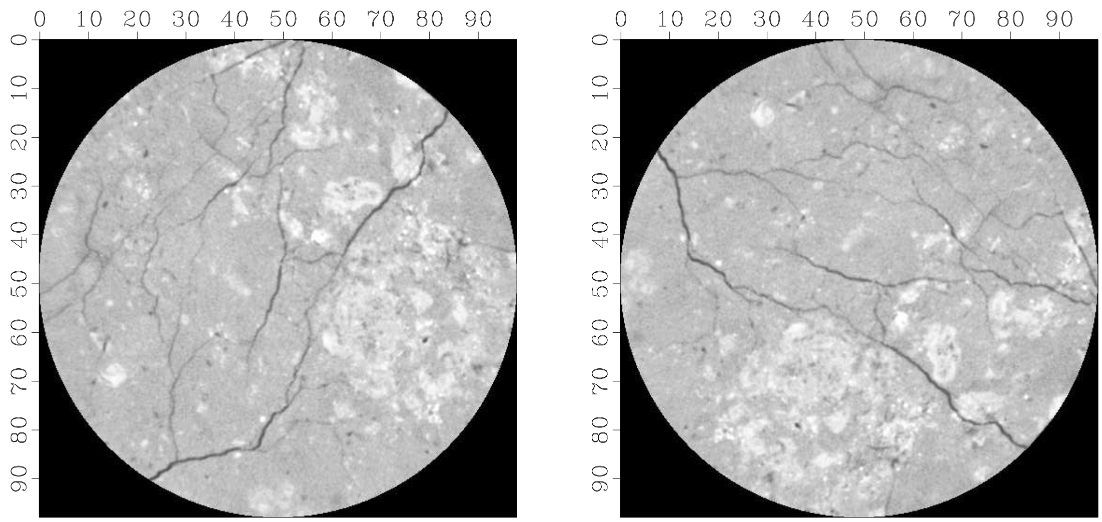
INTERPOLATION AFTER COORDINATE TRANSFORMATION

In this exercise, we will use a slice out of a 3-D CT-scan of a carbonate rock sample, shown in Figure 6a[1]. Notice microfracture channels.

The goal of the exercise is to apply a coordinate transformation to the original data. A particular transformation that we will study is coordinate rotation. Figure 6b shows the original slice rotated by 90 degrees. A 90-degree rotation in this case amounts to simple transpose. However, rotation by a different angle requires interpolation from the original grid to the modified grid.

The task of coordinate rotation is accomplished by the C program `rotate.c`. Two different methods are implemented: nearest-neighbor interpolation and bilinear interpolation.

[1]Courtesy of Jim Jennings (currently at Shell.)



a b

Figure 6: Slice of a CT-scan of a carbonate rock sample. (a) Original. (b) After clockwise rotation by 90°.

To test the accuracy of different methods, we can rotate the original data in small increments and then compare the result of rotating to 360° with the original data. Figure 7 compares the error of the nearest-neighbor and bilinear interpolations after rotating the original slice in increments of 20°. The accuracy is comparatively low for small discontinuous features like microfracture channels.

To improve the accuracy further, we need to employ a longer filter. One popular choice is *cubic convolution* interpolation, invented by Robert Keys (a geophysicist, currently at ConocoPhillips). The cubic convolution filter can be expressed as the filter (Keys, 1981)

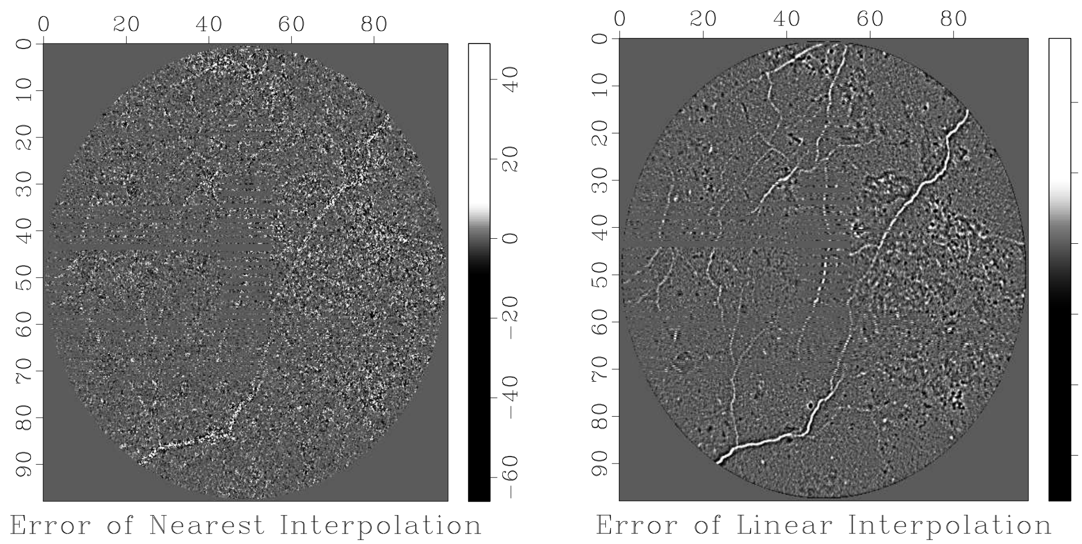
$$Z^\sigma \approx C(Z) = -\frac{\sigma\,(1-\sigma)^2}{2}\,Z^{-1} + \frac{(1-\sigma)\,(2+2\,\sigma-3\sigma^2)}{2} + \frac{\sigma\,(1+4\,\sigma-3\,\sigma^2)}{2}\,Z - \frac{(1-\sigma)\,\sigma^2}{2}\,Z^2\,. \tag{8}$$

and is designed to approximate the ideal sinc-function interpolator.

rotate/rotate.c

```
/* Rotate around. */
#include <rsf.h>

int main(int argc, char* argv[])
{
    int n1, n2, i1, i2, k1, k2;
    float x1, x2, c1, c2, cosa, sina, angle;
    float **orig, **rotd;
```



a b

Figure 7: Error of different interpolation methods computed after full circle rotation in increments of 20 degrees. (a) Nearest-neighbor interpolation. (b) Bi-linear interpolation.

```
    char *interp;
    sf_file inp, out;

    /* initialize */
    sf_init(argc,argv);
    inp = sf_input("in");
    out = sf_output("out");

    /* get dimensions from input */
    if (!sf_histint(inp,"n1",&n1)) sf_error("No n1= in inp");
    if (!sf_histint(inp,"n2",&n2)) sf_error("No n2= in inp");

    /* get parameters from command line */
    if (!sf_getfloat("angle",&angle)) angle=90.;
    /* rotation angle */

    if (NULL == (interp = sf_getstring("interp")))
        interp="nearest";
    /* [n,l,c] interpolation type */

    /* convert degrees to radians */
    angle *= SF_PI/180.;
    cosa = cosf(angle);
```

```
    sina = sinf(angle);

    orig = sf_floatalloc2(n1,n2);
    rotd = sf_floatalloc2(n1,n2);

    /* read data */
    sf_floatread(orig[0],n1*n2,inp);

    /* central point */
    c1 = (n1-1)*0.5;
    c2 = (n2-1)*0.5;

    for (i2=0; i2 < n2; i2++) {
        for (i1=0; i1 < n1; i1++) {

            /* rotated coordinates */
            x1 = c1+(i1-c1)*cosa-(i2-c2)*sina;
            x2 = c2+(i1-c1)*sina+(i2-c2)*cosa;

            /* nearest neighbor */
            k1 = floorf(x1); x1 -= k1;
            k2 = floorf(x2); x2 -= k2;

            switch(interp[0]) {
                case 'n': /* nearest neighbor */
                    if (x1 > 0.5) k1++;
                    if (x2 > 0.5) k2++;
                    if (k1 >=0 && k1 < n1 &&
                        k2 >=0 && k2 < n2) {
                        rotd[i2][i1] = orig[k2][k1];
                    } else {
                        rotd[i2][i1] = 0.;
                    }
                    break;
                case 'l': /* bilinear */
                    if (k1 >=0 && k1 < n1-1 &&
                        k2 >=0 && k2 < n2-1) {
                        rotd[i2][i1] =
                            (1.-x1)*(1.-x2)*orig[k2][k1]   +
                            x1     *(1.-x2)*orig[k2][k1+1] +
                            (1.-x1)*x2     *orig[k2+1][k1] +
                            x1     *x2     *orig[k2+1][k1+1];
                    } else {
                        rotd[i2][i1] = 0.;
                    }
```

```
                break;
            case 'c': /* cubic convolution */
                /* !!! ADD CODE !!! */
                break;
            default:
                sf_error("Unknown interpolation %s",
                         interp);
                break;
        }
    }
}

/* write result */
sf_floatwrite(rotd[0],n1*n2,out);

exit(0);
}
```

rotate/SConstruct

```
from rsf.proj import *

# Download data
Fetch('slice.rsf','ctscan')
Flow('circle','slice','dd type=float')

grey = 'grey wanttitle=n screenratio=1 bias=128 clip=105'

Result('circle',grey)

# Rotate program
program = Program('rotate.c')
rotate = str(program[0])

# Rotate by 90 degrees
Flow('rotate',['circle',rotate],
     './${SOURCES[1]} angle=90 interp=nearest')

Result('rotate',grey)

# Mask for the circle
Flow('mask','circle',
     '''
     put d1=1 o1=-255.5 d2=1 o2=-255.5 |
     math output="sqrt(x1*x1+x2*x2)" |
```

```
      mask min=255.5 | dd type=float |
      smooth rect1=3 rect2=3 |
      mask max=0 | dd type=float |
      put d1=0.1914 o1=0 d2=0.1914 o2=0
      ''')

for case in ('nearest','linear'): # !!! MODIFY ME !!!
    new = 'circle'
    rotates = []
    for r in range(18):
        old = new
        new = '%s-circle%d' % (case,r)
        Flow(new,[old,rotate],
             './${SOURCES[1]} angle=20 interp=%s' % case)
        Plot(new,grey)
        rotates.append(new)

    # Movie of rotating circle
    Plot(case,rotates,'Movie',view=1)

    # Plot error
    Result(case,[new,'circle','mask'],
           '''
           add scale=1,-1 ${SOURCES[1]} |
           add mode=p ${SOURCES[2]} |
           %s bias=0 scalebar=y clip=12
           wanttitle=y title="Error of %s Interpolation"
           ''' % (grey,case.capitalize()))

End()
```

Your task:

1. Change directory to `hw4/rotate`
2. Run

   `scons view`

   to reproduce the figures on your screen.
3. Additionally, you can run

   `scons nearest.vpl`

   and

```
scons linear.vpl
```

to see movies of incremental slice rotation with different methods.

4. Modify the `rotate.c` program and the `SConstruct` file to implement the cubic convolution interpolation and to compare its results with the two other methods.

5. **EXTRA CREDIT** for implementing an interpolation algorithm, which is more accurate than cubic convolution.

YOUR OWN DATA

Your final task is to apply one of the data analysis techniques of the previous sections (Fourier compression or coordinate transformation) to your own data:

1. Select a dataset suitable for compression or coordinate transformation.

2. Apply one of the algorithms of the previous two sections and choose appropriate parameters.

3. Include the results in your homework.

COMPLETING THE ASSIGNMENT

1. Change directory to `hw3`.

2. Edit the file `paper.tex` in your favorite editor and change the first line to have your name instead of Fourier's.

3. Run

   ```
   sftour scons lock
   ```

   to update all figures.

4. Run

   ```
   sftour scons -c
   ```

   to remove intermediate files.

5. Run

   ```
   scons pdf
   ```

   to create the final document.

6. Submit your result (file `paper.pdf`) on paper or by e-mail.

REFERENCES

Hall, M., 2007, Smooth operator: Smoothing seismic interpretations and attributes: The Leading Edge, **26**, 16–20.

Keys, R. G., 1981, Cubic convolution interpolation for digital image processing: IEEE Transactions on Acoustics, Speech, and Signal Processing, **ASSP-29**, 1153–1160.