Asymptotic pseudounitary stacking operators[a]

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ABSTRACT

Stacking operators are widely used in seismic imaging and seismic data processing. Examples include Kirchhoff datuming, migration, offset continuation, DMO, and velocity transform. Two primary approaches exist for inverting such operators. The first approach is iterative least-squares optimization, which involves the construction of the adjoint operator. The second approach is asymptotic inversion, where an approximate inverse operator is constructed in the high-frequency asymptotics. Adjoint and asymptotic inverse operators share the same kinematic properties, but their amplitudes (weighting functions) are defined differently. This paper describes a theory for reconciling the two approaches. I introduce a pair of the *asymptotic pseudo-unitary* operators, which possess both the property of being adjoint and the property of being asymptotically inverse. The weighting function of the asymptotic pseudo-unitary stacking operators is shown to be completely defined by the derivatives of the operator kinematics. I exemplify the general theory by considering several particular examples of stacking operators. Simple numerical experiments demonstrate a noticeable gain in efficiency when the asymptotic pseudo-unitary operators are applied for preconditioning iterative least-squares optimization.

INTRODUCTION

Integral (stacking) operators play an important role in seismic imaging and seismic data processing. The most common applications are common midpoint stacking, Kirchhoff migration, and dip moveout. Other examples include (listed in random order) Kirchhoff datuming, back-projection tomography, slant stack, velocity transform, offset continuation, and azimuth moveout. The use of the integral methods increases in prestack three-dimensional processing because of their flexibility with respect to irregularities in the data geometry.

An integral operator often is used to represent the forward modeling problem, and we invert it to solve for the model. In this paper, I consider two different approaches to inversion. The first is least-squares inversion, which requires constructing the adjoint counterpart of the modeling operator. The second approach is asymptotic inversion,

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which aims at reconstructing the high-frequency (discontinuous) parts of the model. I compare the two approaches and introduce the notion of *asymptotic pseudo-unitary* operator pair that ties them together.

In practice, least squares inversion is often applied as an iterative process (Ronen and Liner, 2000). The advantage of connecting it with the asymptotic inverse theory is the ability to speed up the iteration. This approach was used, in the context of seismic migration, by Jin et al. (1992) and Lambaré et al. (1992). Asymptotic pseudo-unitary operators, introduced in this paper, provide a more universal theoretical tool. One can use them to construct an appropriate preconditioning operator for accelerating the convergence of the least-squares methods.

The first part of this paper contains a formal definition of a stacking operator and reviews the theory of asymptotic inversion, following the fundamental results of Beylkin (1985) and Goldin (1988, 1990). According to this theory, the high-frequency asymptotic inverse of a stacking operator is also a stacking operator. To connect this theory with the theory of adjoint operators, I show that the adjoint of a stacking operator can also be included in the class of stacking operators. The adjoint operator has the same summation path as the asymptotic inverse but a different weighting function. These two results combine together to form the definition of asymptotic pseudo-unitary integral operators. I apply such operators to define a general preconditioning operator for least-squares inversion. While one can apply Beylkin's theory directly for constructing an appropriate asymptotic preconditioner, pseudo-unitary operators accomplish the job in a more straightforward and computationally attractive way.

The second part of the paper addresses such examples of commonly used stacking operators as wave-equation datuming, migration, velocity transform, and offset continuation. The theory is specified for these particular applications and accompanied by numerical examples. The examples demonstrate the practical advantages of asymptotic pseudo-unitary operators.

THEORETICAL DEFINITION OF A STACKING OPERATOR

In practice, integration of discrete data is performed by stacking. In theory, it is convenient to represent a stacking operator in the form of a continuous integral:

$$S(t,y) = \mathbf{A}\left[M(z,x)\right] = \int\limits_{\Omega} w(x;t,y)\, M(\theta(x;t,y),x)\, dx\ . \qquad (1)$$

Function $M(z,x)$ is the input of the operator, $S(t,y)$ is the output, Ω is the summation aperture, θ represents the summation path, and w stands for the weighting function. The range of integration (the operator aperture) may also depend on t and y. Allowing x to be a two-dimensional variable, we can use definition (1) to represent an operator applied to three-dimensional data. Throughout this paper, I assume that t and z belong to a one-dimensional space, and that x and y have the same number of dimensions.

The goal of inversion is to reconstruct some function $\widehat{M}(z,x)$ for a given $S(t,y)$, so that $\widehat{M}$ is in some sense close to M in equation (1).

ASYMPTOTIC INVERSION: RECONSTRUCTING THE DISCONTINUITIES

Mathematical analysis of the inverse problem for operator (1) shows that only in rare cases can we obtain an analytically exact inversion. A well-known example is the Radon transform, which has acquired a lot of different aliases in geophysical literature: slant stack, tau-p transform, plane wave decomposition, and controlled directional reception (CDR) transform (Gardner and Lu, 1991). In this case,

$$\theta(x;t,y) = t + x\,y\;, \qquad (2)$$

$$w(x;t,y) = 1\;. \qquad (3)$$

Radon obtained a result similar to the theoretical inversion of operator (1) with the summation path (2) and the weighting function (3) in 1917, but his result was not widely known until the development of computer tomography. According to Radon (1917), the inverse operator has the form

$$M(z,x) = \mathbf{A^{-1}}[S(t,y)] = |\mathbf{D}|^m \int \widehat{w}\,S(\widehat{\theta}(y;z,x),y)\,dy\;, \qquad (4)$$

where

$$\widehat{\theta}(y;z,x) = z - x\,y\;, \qquad (5)$$

$$\widehat{w} = \frac{1}{(2\,\pi)^m}\;, \qquad (6)$$

$|\mathbf{D}|$ is a one-dimensional convolution operator with the spectrum $|\omega|$:

$$|\mathbf{D}|\,[U(z,x)] = \frac{1}{2\,\pi}\int U(\xi,x)\int |\omega|\,e^{i\omega(z-\xi)}d\omega\,d\xi\;, \qquad (7)$$

and m is the dimensionality of x and y (usually 1 or 2). In Russian geophysical literature, a similar result for the inversion of the CDR transform was published by Nakhamkin (1969).

Extension of Radon's result to the general form of integral operator (1) (*generalized Radon transform*) is possible via asymptotic analysis of the inverse problem. In the general case, Beylkin (1985) and Goldin (1988) have shown that asymptotic inversion can reconstruct discontinuous parts of the model. These are the parts responsible for the asymptotic behavior of the model at high frequencies. Since the discontinuities are associated with wavefronts and reflection events at seismic sections, there is a certain correspondence between asymptotic inversion and such standard goals of seismic data processing as kinematic equivalence and amplitude preservation.

The main theorem of asymptotic inversion can be formulated as follows (Goldin, 1988). The leading-order discontinuities in M are reconstructed by an integral operator of the form

$$\widehat{M}(z,x) = \widehat{\mathbf{A}}[S(t,y)] = |\mathbf{D}|^m \int \widehat{w}(y;z,x)\, S(\widehat{\theta}(y;z,x),y)\, dy \;, \tag{8}$$

where the summation path $\widehat{\theta}$ is obtained simply by solving the equation

$$z = \theta(x;t,y) \tag{9}$$

for t (if such an explicit solution is possible). The correctly chosen summation path reconstructs the geometry of the discontinuities. To recover the amplitude, we must choose the correct weighting function, which is constrained by the equation (Beylkin, 1985; Goldin, 1988)

$$w\,\widehat{w} = \frac{1}{(2\,\pi)^m} \sqrt{\left|F\,\widehat{F}\right| \left|\frac{\partial\widehat{\theta}}{\partial z}\right|^m} \;, \tag{10}$$

where

$$F = \frac{\partial\theta}{\partial t}\,\frac{\partial^2\theta}{\partial x\,\partial y} - \frac{\partial\theta}{\partial y}\,\frac{\partial^2\theta}{\partial x\,\partial t} \;, \tag{11}$$

$$\widehat{F} = \frac{\partial\widehat{\theta}}{\partial z}\,\frac{\partial^2\widehat{\theta}}{\partial x\,\partial y} - \frac{\partial\widehat{\theta}}{\partial x}\,\frac{\partial^2\widehat{\theta}}{\partial y\,\partial z} \;. \tag{12}$$

The solution assumes that differential forms F and $\widehat{F}$ exist and are bounded and non-vanishing[2]. In the multi-dimensional case ($m \geq 2$), they are replaced by the determinants of the corresponding matrices. To ensure the asymptotic inversion, equation (10) must be satisfied at least in the vicinity of the *stationary points* of integral (1). Those are the points where the summation path of the form (9) is tangent to the traveltimes of the actual events on the transformed model. In the case of the Radon transform, $\left|F\,\widehat{F}\right| = \left|\frac{\partial\widehat{\theta}}{\partial z}\right| = 1$, and the asymptotic inverse coincides with the exact inversion.

LEAST-SQUARES INVERSION AND ADJOINT OPERATORS

Least-squares inversion is widely used in practice not only because it is applicable even when the asymptotic results are unavailable but also because of its ability to

[2]This requirement is related to the requirement for the normal $\mathbf{A}^T\,\mathbf{A}$ operator, inroduced in the next section, to be a pseudo-differential operator (Wong, 1991). Situations where this condition is violated require a special consideration (Nolan and Symes, 1996; Stolk, 2000).

handle finite sampling effects that are difficult to handle in asymptotic theory (Ronen and Liner, 2000).

The theoretical least-squares inverse of operator (1) has the well-known form (Tarantola, 1987)

$$\widetilde{M}(z,x) = \widetilde{\mathbf{A}}[S(t,y)] = \left(\mathbf{A^T}\,\mathbf{A}\right)^{\dagger}\,\mathbf{A^T}[S(t,y)]\;, \tag{13}$$

where † denotes pseudo-inverse, and the adjoint operator $\mathbf{A^T}$ is defined by the dot-product test:

$$(S(t,y), \mathbf{A}[M(z,x)]) \equiv \left(\mathbf{A^T}[S(t,y)], M(z,x)\right)\;. \tag{14}$$

With a specified definition of the dot-product, the generalized inverse minimizes the following quantity, which is the squared L_2 norm of the residual:

$$(S(t,y) - \mathbf{A}[M(z,x)], S(t,y) - \mathbf{A}[M(z,x)])\;. \tag{15}$$

In the case of integral operators, a natural definition of the dot-product is the double integral

$$(S_1(t,y), S_2(t,y)) = \iint S_1(t,y)\,S_2(t,y)\,dy\,dt\;, \tag{16}$$

$$(M_1(z,x), M_2(z,x)) = \iint M_1(z,x)\,M_2(z,x)\,dx\,dz\;. \tag{17}$$

The notion of the adjoint operator completely depends on the arbitrarily chosen definition of the dot product and norm in the model and data spaces. A simple way to change those definitions is to find some positive weights $W_M(z,x)$ in the model space and $W_S(t,y)$ in the data space that define the dot products as follows:

$$(S_1(t,y), S_2(t,y)) = \iint W_S(t,y)\,S_1(t,y)\,S_2(t,y)\,dy\,dt\;, \tag{18}$$

$$(M_1(z,x), M_2(z,x)) = \iint W_M(z,x)\,M_1(z,x)\,M_2(z,x)\,dx\,dz\;. \tag{19}$$

To formally define the adjoint of a stacking operator, let us substitute the definition of the stacking operator (1) into the dot product (14), as follows:

$$(S(t,y), \mathbf{A}[M(z,x)]) = \int\iint w(x;t,y)\,M(\theta(x;t,y),x)\,S(t,y)\,dx\,dy\,dt\;. \tag{20}$$

Assuming that the function θ is monotone in t [3], we can change the integration variable t to $z = \theta(x;t,y)$ and rewrite equation (20) in the form

$$(S(t,y), \mathbf{A}[M(z,x)]) = \int\iint \widetilde{w}(y;z,x)\,M(z,x)\,S(\widehat{\theta}(y;z,x),x)\,dy\,dx\,dz\;, \tag{21}$$

[3] If this is not the case, a different parameterization of the stacking function is appropriate (Fomel, 2001a)

where $\widehat{\theta}$ has the same meaning as in equation (8), and

$$\widetilde{w}(y;z,x) = w(x;\widehat{\theta}(y;z,x),y) \left|\frac{\partial\widehat{\theta}}{\partial z}\right| . \tag{22}$$

Comparing equations (21) and (14), we conclude that the adjoint operator $\mathbf{A^T}$ is defined by the equality

$$\mathbf{A^T}[S(t,y)] = \int \widetilde{w}(y;z,x)\,S(\widehat{\theta}(y;z,x),y)\;dy . \tag{23}$$

Thus we have proven that the continuous adjoint of a stacking operator is another stacking operator. The adjoint operator has the same summation path as the asymptotic inverse (8), which guarantees the correct reconstruction of the kinematics of the input wavefield. The amplitude (weighting function) of the adjoint operator is directly proportional to the forward weighting according to equation (22). The coefficient of proportionality is the Jacobian of the transformation of the variables z and t.

Similar results have been obtained for particular cases of stacking operators: velocity transform (Thorson, 1984; Jedlicka, 1989), Kirchhoff constant-velocity migration (Ji, 1994), and NMO (Crawley, 1995). In the appendix, I exemplify an application of least-squares inversion by reviewing inversion of the Radon operator and showing that it is precisely equivalent to the asymptotic result of the previous section.

ASYMPTOTIC PSEUDO-UNITARY OPERATOR PAIR

According to the theory of asymptotic inversion, briefly reviewed in the first part of this paper, the weighting function of the asymptotically inverse operator is *inversely* proportional to the weighting of the forward operator. On the other hand, the weighting in the adjoint is *directly* proportional to the forward weighting. This difference allows us to define a hybrid pair of operators that possess both the property of being adjoint and the property of being asymptotic inverse. It is appropriate to call a pair of operators defined in this way *asymptotic pseudo-unitary*. The definition of asymptotic pseudo-unitary operators follows directly from the combination of definitions (8) and (23). Splitting the derivative operator $|\mathbf{D}|$ in (8) into the product of two operators, we can write the forward operator as

$$S(t,y) = \mathbf{A}\left[M(z,x)\right] = \int w^{(+)}(x;t,y)\,|\mathbf{D}|^{m/2}\,M(\theta(x;t,y),x)\,dx \tag{24}$$

and its asymptotic pseudo-unitary adjoint as

$$\widetilde{M}(z,x) = \widetilde{\mathbf{A}}[S(t,y)] = |\mathbf{D}|^{m/2} \int w^{(-)}(y;z,x)\,S(\widehat{\theta}(y;z,x),y)\;dy . \tag{25}$$

According to equation (10),

$$w^{(+)}\,w^{(-)} = \frac{1}{(2\,\pi)^m}\,\sqrt{\left|F\,\widehat{F}\right|\,\left|\frac{\partial\widehat{\theta}}{\partial z}\right|^m}\;. \tag{26}$$

According to equation (22),

$$w^{(-)} = w^{(+)}\,\left|\frac{\partial\widehat{\theta}}{\partial z}\right|\;. \tag{27}$$

Combining equations (26) and (27) uniquely determines both weighting functions, as follows:

$$w^{(+)} = \frac{1}{(2\,\pi)^{m/2}}\,\left|F\,\widehat{F}\right|^{1/4}\,\left|\frac{\partial\widehat{\theta}}{\partial z}\right|^{(m-2)/4}\;, \tag{28}$$

$$w^{(-)} = \frac{1}{(2\,\pi)^{m/2}}\,\left|F\,\widehat{F}\right|^{1/4}\,\left|\frac{\partial\widehat{\theta}}{\partial z}\right|^{(m+2)/4}\;. \tag{29}$$

Equations (28) and (29) complete the definition of asymptotic pseudo-unitary operator pair.

The notion of pseudo-unitary operators is directly applicable in the situations where we can arbitrarily construct both forward and inverse operators. One example of such a situation is the velocity transform considered in the next section of this paper. In the more common case, the forward operator is strictly defined by the physics of a problem. In this case, we can include asymptotic inversion in the iterative least-squares inversion by means of *preconditioning* (Jin et al., 1992; Lambaré et al., 1992). The linear preconditioning operator should transform the forward stacking-type operator to the form (24) with the weighting function (28). Theoretically, this form of preconditioning should lead to the fastest convergence of the iterative least-squares inversion with respect to the high-frequency parts of the model.

If the forward pseudo-unitary operator $\mathbf{A}_p$ can be related to the forward modeling operator $\mathbf{A}_m$ as $\mathbf{A}_p = \mathbf{W}_s\,\mathbf{A}_m\,\mathbf{W}_m$, where $\mathbf{W}_s$ and $\mathbf{W}_m$ are weighting operators in the data and model domains correspondingly, then preconditioning simply amounts to replacing the least-squares equation

$$S \approx \mathbf{A}_m[M] \tag{30}$$

with the equation

$$\mathbf{W}_s[S] \approx \mathbf{W}_s\,\mathbf{A}_m\,\mathbf{W}_m[P] = \mathbf{A}_p[P]\;, \tag{31}$$

where P is the preconditioned model. The advantage of using equation (31) is in the the fact that the normal operator $\mathbf{A}_p^T\,\mathbf{A}_p$ is closer (asymptotically) to identity and therefore should be easier to invert than the original operator $\mathbf{A}_m^T\,\mathbf{A}_m$ in the least-squares solution (13).

EXAMPLES

In this section, I consider several particular examples of stacking operators used in seismic data processing and derive their asymptotic pseudo-unitary versions.

Datuming

Let x denote a point on the surface at which the propagating wavefield is recorded. Let y denote a point on another surface, to which the wavefield is propagating. Then the summation path of the stacking operator for the forward wavefield continuation is

$$\theta(x;t,y) = t - T(x,y) \; , \tag{32}$$

where t is the time recorded at the y-surface, and $T(x,y)$ is the traveltime along the ray connecting x and y. The backward propagation reverses the sign in (32), as follows:

$$\widehat{\theta}(y;z,x) = z + T(x,y) \; . \tag{33}$$

Substituting the summation path formulas (32) and (33) into the general weighting function formulas (28) and (29), we immediately obtain

$$w^{(+)} = w^{(-)} = \frac{1}{(2\,\pi)^{m/2}} \left|\frac{\partial^2 T}{\partial x\,\partial y}\right|^{1/2} \; . \tag{34}$$

Gritsenko's formula (Gritsenko, 1984; Goldin, 1986) states that the second mixed traveltime derivative $\frac{\partial^2 T}{\partial x\,\partial y}$ is connected with the geometric spreading R along the x-y ray by the equality

$$R(x,y) = \frac{\sqrt{\cos\alpha(x)\,\cos\alpha(y)}}{v(x)} \left|\frac{\partial^2 T}{\partial x\,\partial y}\right|^{-1/2} \; , \tag{35}$$

where $v(x)$ is the velocity at the point x, and $\alpha(x)$ and $\alpha(y)$ are the angles formed by the ray with the x and y surfaces, respectively. In a constant-velocity medium,

$$R(x,y) = v^{m-1}\,T(x,y)^{m/2} \; . \tag{36}$$

Gritsenko's formula (35) allows us to rewrite equation (34) in the form (Goldin, 1988)

$$w^{(+)}(x;t,y) = \frac{1}{(2\,\pi)^{m/2}} \frac{\sqrt{\cos\alpha(x)\,\cos\alpha(y)}}{v(x)\,R(x,y)} \; , \tag{37}$$

$$w^{(-)}(y;z,x) = \frac{1}{(2\,\pi)^{m/2}} \frac{\sqrt{\cos\alpha(x)\,\cos\alpha(y)}}{v(y)\,R(y,x)} \; . \tag{38}$$

The weighting functions commonly used in Kirchhoff datuming (Berryhill, 1979; Wiggins, 1984; Goldin, 1985) are defined as

$$w(x;t,y) = \frac{1}{(2\,\pi)^{m/2}}\,\frac{\cos\alpha(x)}{v(x)\,R(x,y)}\;, \tag{39}$$

$$\widehat{w}(y;z,x) = \frac{1}{(2\,\pi)^{m/2}}\,\frac{\cos\alpha(y)}{v(y)\,R(y,x)}\;. \tag{40}$$

These two operators appear to be asymptotically inverse according to formula (10). They coincide with the asymptotic pseudo-unitary operators if the velocity v is constant ($v(x) = v(y)$), and the two datum surfaces are parallel ($\alpha(x) = \alpha(y)$).

Migration

Least-squares migration, envisioned by Lailly (1984) and Tarantola (1984), has recently become a practical method and gained a lot of attention in the geophysical literature (Nemeth et al., 1999; Chavent and Plessix, 1999; Duquet and Marfurt, 1999; Fomel et al., 2002). Using the theory of asymptotic pseudo-unitary operators allows us to reconcile this approach with the method of *asymptotic true-amplitude* migration (Bleistein et al., 2001).

As recognized by Tygel et al. (1994), true-amplitude migration (Goldin, 1992; Schleicher et al., 1993) is the asymptotic inversion of seismic modeling represented by the Kirchhoff high-frequency approximation. The Kirchhoff approximation for a reflected wave (Haddon and Buchen, 1981; Bleistein, 1984) belongs to the class of stacking-type operators (1) with the summation path

$$\theta(x;t,y) = t - T\,(s(y),x) - T\,(x,r(y))\;\;, \tag{41}$$

the weighting function

$$w(x;t,y) = \frac{1}{(2\,\pi)^{m/2}}\,\frac{C\,(s(y),x,r(y))}{R\,(s(y),x)\;R\,(x,r(y))}\;, \tag{42}$$

and the additional time filter $\left(\frac{\partial}{\partial z}\right)^{m/2}$. Here x denotes a point at the reflector surface, s is the source location, and r is the receiver location at the observation surface. The parameter y corresponds to the configuration of observation. That is, $s(y) = s\,,\ r(y) = y$ for the common-shot configuration, $s(y) = r(y) = y$ for the zero-offset configuration, and $s(y) = y - h\,,\ r(y) = y + h$ for the common-offset configuration (where h is the half-offset). The functions T and R have the same meaning as in the datuming example, representing the one-way traveltime and the one-way geometric spreading, respectively. The function $C(s,x,r)$ is known as the *obliquity factor*. Its definition is

$$C(s,x,r) = \frac{1}{2}\left(\frac{\cos\alpha_s(x)}{v_s(x)} + \frac{\cos\alpha_r(x)}{v_r(x)}\right)\;, \tag{43}$$

where the angles $\alpha_s(x)$ and $\alpha_r(x)$ are formed by the incident and reflected waves with the normal to the reflector at the point x, and $v_s(x)$ and $v_r(x)$ are the corresponding velocities in the vicinity of this point. In this paper, I leave the case of converted (e.g., P-SV) waves outside the scope of consideration and assume that $v_s(x)$ equals $v_r(x)$ (e.g., in P-P reflection). In this case, it is important to notice that at the stationary point of the Kirchhoff integral, $\alpha_s(x) = \alpha_r(x) = \alpha(x)$ (the law of reflection), and therefore

$$C(s,x,r) = \frac{\cos\alpha(x)}{v(x)} \; . \tag{44}$$

The stationary point of the Kirchhoff integral is the point where the stacking curve (41) is tangent to the actual reflection traveltime curve. When our goal is asymptotic inversion, it is appropriate to use equation (44) in place of (43) to construct the inverse operator. The weighted function (42) can include other factors affecting the leading-order (WKBJ) ray amplitude, such as the source signature, caustics counter (the KMAH-index), and transmission coefficient for the interfaces (Chapman and Drummond, 1982; Červený, 2001). In the following analysis, I neglect these factors for simplicity.

The model M implied by the Kirchhoff modeling integral is the wavefield with the wavelet shape of the incident wave and the amplitude proportional to the reflector coefficient along the reflector surface. The goal of true-amplitude migration is to recover M from the observed seismic data. In order to obtain the image of the reflectors, the reconstructed model is evaluated at the time z equal to zero. The Kirchhoff modeling integral requires explicit definition of the reflector surface. However, its inverse doesn't require explicit specification of the reflector location. For each point of the subsurface, one can find the normal to the hypothetical reflector by bisecting the angle between the $s-x$ and $x-r$ rays. Born scattering approximation provides a different physical model for the reflected waves. According to this approximation, the recorded waves are viewed as scattered on smooth local inhomogeneities rather than reflected from sharp reflector surfaces. The inversion of Born modeling (Miller et al., 1987; Bleistein, 1987) closely corresponds with the result of Kirchhoff integral inversion. For an unknown reflector and the correct macro-velocity model, the asymptotic inversion reconstructs the signal located at the reflector surface with the amplitude proportional to the reflector coefficient.

As follows from the form of the summation path (41), the integral migration operator must have the summation path

$$\widehat{\theta}(y;z,x) = z + T\left(s(y),x\right) + T\left(x,r(y)\right) \tag{45}$$

to reconstruct the geometry of the reflector at the migrated section. According to equation (8), the asymptotic reconstruction of the wavelet requires, in addition, the derivative filter $\left(-\frac{\partial}{\partial t}\right)^{m/2}$. The asymptotic reconstruction of the amplitude defines the true-amplitude weighting function in accordance with equation (10), as follows:

$$\widehat{w}(y;z,x) = \frac{v(x)\,R\left(s(y),x\right)\;R\left(x,r(y)\right)}{(2\,\pi)^{m/2}\;\cos\alpha(x)} \left|\frac{\partial^2 T\left(s(y),x\right)}{\partial x\,\partial y} + \frac{\partial^2 T\left(x,r(y)\right)}{\partial x\,\partial y}\right| \; . \tag{46}$$

The weighting function of the asymptotic pseudo-unitary migration is found analogously to equation (34) as

$$w^{(+)} = w^{(-)} = \frac{1}{(2\,\pi)^{m/2}} \left| \frac{\partial^2 T\,(s(y),x)}{\partial x\,\partial y} + \frac{\partial^2 T\,(x,r(y))}{\partial x\,\partial y} \right|^{1/2} . \tag{47}$$

Unlike true-amplitude migration, this type of migration operator does not change the dimensionality of the input. Several specific cases exist for different configurations of the input data.

1. Common-shot migration

In the case of common-shot migration, we can simplify equation (46) with the help of Gritsenko's formula (35) to the form

$$\widehat{w}_{CS}(r;z,x) = \frac{1}{(2\,\pi)^{m/2}}\,\frac{\cos\alpha(r)}{v(x)}\,\frac{R(s,x)}{R(x,r)} = \frac{1}{(2\,\pi)^{m/2}}\,\frac{\cos\alpha(r)}{v(r)}\,\frac{R(s,x)}{R(r,x)}\;, \tag{48}$$

where the angle $\alpha(r)$ is measured between the reflected ray and the normal to the observation surface at the reflector point r. Formula (48) coincides with the analogous result of Keho and Beydoun (1988), derived directly from Claerbout's imaging principle (Claerbout, 1970). An alternative derivation is given by Goldin (1987). Docherty (1991) points out a remarkable correspondence between this formula and the classic results of Born scattering inversion (Bleistein, 1987).

For common-shot migration, pseudo-unitary weighting coincides with the weighting of datuming and corresponds to the downward continuation of the receivers.

2. Zero-offset migration

In the case of zero-offset migration, Gritsenko's formula simplifies the true-amplitude migration weighting function (46) to the form

$$\widehat{w}_{ZO}(y;z,x) = \frac{2^m}{(2\,\pi)^{m/2}}\,\frac{\cos\alpha(y)}{v(y)}\;. \tag{49}$$

In a constant-velocity medium, one can accomplish the true-amplitude zero-offset migration by premultiplying the recorded zero-offset seismic section by the factor $\left(\frac{v}{2}\right)^{m-1}\left(\frac{t}{2}\right)^{m/2}$ [which corresponds at the stationary point to the geometric spreading $R(x,y)$] and downward continuation according to formula (40) with the effective velocity $v/2$ (Goldin, 1987; Hubral et al., 1991). This conclusion is in agreement with the analogous result of Born inversion (Bleistein et al., 1985), though derived from a different viewpoint.

In the zero-offset case, the pseudo-unitary forward operator reduces to downward pseudo-unitary continuation with a velocity of $v/2$.

3. Common-offset migration

In the case of common-offset migration in a general variable-velocity medium, the weighting function (46) cannot be simplified to a different form, and all its components need to be calculated explicitly by dynamic ray tracing (Červený and de Castro, 1993). In the constant-velocity case, we can differentiate the explicit expression for the summation path

$$\widehat{\theta}(y;z,x) = z + \frac{\rho_s(x,y) + \rho_r(x,y)}{v} \;, \tag{50}$$

where ρ_s and ρ_r are the lengths of the incident and reflected rays:

$$\rho_s(y,x) = \sqrt{x_3^2 + (x_1 - y_1 + h_1)^2 + (x_2 - y_2 + h_2)^2} \;, \tag{51}$$

$$\rho_r(y,x) = \sqrt{x_3^2 + (x_1 - y_1 - h_1)^2 + (x_2 - y_2 - h_2)^2} \;. \tag{52}$$

For simplicity, the vertical component of the midpoint y_3 is set here to zero. Evaluating the second derivative term in formula (46) for the common-offset geometry leads, after some heavy algebra, to the expression

$$\left|\frac{\partial^2 T\,(s(y),x)}{\partial x\,\partial y} + \frac{\partial^2 T\,(x,r(y))}{\partial x\,\partial y}\right| = \frac{x_3\,(\rho_s^2+\rho_r^2)}{v\,(\rho_s\,\rho_r)^2}\,\left(\frac{\rho_s+\rho_r}{v\,\rho_s\,\rho_r}\right)^{m-1}\,\cos\alpha(x)\;. \tag{53}$$

Substituting (53) into the general formula (46) yields the weighting function for the common-offset true-amplitude constant-velocity migration:

$$\widehat{w}_{CO}(y;z,x) = \frac{1}{(2\,\pi)^{m/2}}\,\frac{x_3\,(\rho_s+\rho_r)^{m-1}\,(\rho_s^2+\rho_r^2)}{v\,(\rho_s\,\rho_r)^{m/2+1}}\;. \tag{54}$$

Equation (54) is similar to the result obtained by Sullivan and Cohen (1987). In the case of zero offset $h = 0$, it reduces to equation (49). Note that the value of $m = 1$ in (54) corresponds to the two-dimensional (cylindric) waves recorded on the seismic line. A special case is the 2.5-D inversion, when the waves are assumed to be spherical, while the recording is on a line, and the medium has cylindric symmetry. In this case, the modeling weighting function (42) transforms to (Deregowski and Brown, 1983; Bleistein, 1986)

$$w(x;t,y) = \frac{1}{(2\,\pi)^{1/2}}\,\frac{\sqrt{v}\,C\,(s(y),x,r(y))}{\sqrt{\rho_s\,\rho_r\,(\rho_s+\rho_r)}}\;, \tag{55}$$

and the time filter is $\left(\frac{\partial}{\partial z}\right)^{1/2}$. Combining this result with formula (53) for $m = 1$, we obtain the weighting function for the 2.5-D common-offset migration in a constant velocity medium (Sullivan and Cohen, 1987):

$$\widehat{w}_{CO;2.5D}(y;z,x) = \frac{1}{(2\,\pi)^{1/2}}\,\frac{x_3\,\sqrt{\rho_s+\rho_r}\,(\rho_s^2+\rho_r^2)}{\sqrt{v}\,(\rho_s\,\rho_r)^{3/2}}\;. \tag{56}$$

The corresponding time filter for 2.5-D migration is $\left(-\frac{\partial}{\partial t}\right)^{1/2}$.

In the common-offset case, the pseudo-unitary weighting is defined from (47) and (53) as follows:

$$w_{CO}^{(-)}(y;z,x) = \frac{1}{(2\,\pi\,v)^{m/2}}\,\frac{\sqrt{x_3\,\cos\alpha}\,(\rho_s+\rho_r)^{\frac{m-1}{2}}\,\sqrt{\rho_s^2+\rho_r^2}}{(\rho_s\,\rho_r)^{\frac{m+1}{2}}}\;, \tag{57}$$

where

$$\cos\alpha = \left(\frac{(x-y)^2+\rho_s\,\rho_r-h^2}{2\,\rho_s\,\rho_r}\right)^{1/2}\;. \tag{58}$$

Post-Stack Time Migration

An interesting example of a stacking operator is the hyperbola summation used for time migration in the post-stack domain. In this case, the summation path is defined as

$$\widehat{\theta}(y;z,x) = \sqrt{z^2+\frac{(x-y)^2}{v^2}}\;, \tag{59}$$

where z denotes the vertical traveltime, x and y are the horizontal coordinates on the migrated and unmigrated sections respectively, and v stands for the effectively constant root-mean-square velocity (Claerbout, 1995). The summation path for the reverse transformation (demigration) is found from solving equation (59) for z. It has the well-known elliptic form

$$\theta(x;t,y) = \sqrt{t^2-\frac{(x-y)^2}{v^2}}\;. \tag{60}$$

The Jacobian of transforming z to t is

$$\left|\frac{\partial\widehat{\theta}}{\partial z}\right| = \frac{z}{t}\;. \tag{61}$$

If the migration weighting function is defined by conventional downward continuation (Schneider, 1978), it takes the following form, which is equivalent to equation (40):

$$\widehat{w}(y;z,x) = \frac{1}{(2\,\pi)^{m/2}}\,\frac{\cos\alpha(y)}{v\,R(y,x)} = \frac{1}{(2\,\pi)^{m/2}}\,\frac{\cos\alpha}{v^m\,t^{m/2}}\;. \tag{62}$$

The simple trigonometry of the reflected ray suggests that the cosine factor in formula (62) is equal to the simple ratio between the vertical traveltime z and the zero-offset reflected traveltime t:

$$\cos\alpha = \frac{z}{t}\;. \tag{63}$$

The equivalence of the Jacobian (61) and the cosine factor (63) has important interpretations in the theory of Stolt frequency-domain migration (Stolt, 1978; Chun and

Jacewitz, 1981; Levin, 1986). According to equation (22), the weighting function of the adjoint operator is the ratio of (62) and (61):

$$\widetilde{w}(x;t,y) = \frac{1}{(2\,\pi)^{m/2}}\,\frac{1}{v^m\,t^{m/2}}\;. \tag{64}$$

We can see that the cosine factor z/t disappears from the adjoint weighting. This is completely analogous to the known effect of "dropping the Jacobian" in Stolt migration (Harlan, 1983; Levin, 1994). The product of the weighting functions for the time migration and its asymptotic inverse is defined according to formula (10) as

$$w\,\widehat{w} = \frac{1}{(2\,\pi)^m}\,\sqrt{\left|F\,\widehat{F}\right|\,\left|\frac{\partial\widehat{\theta}}{\partial z}\right|^m} = \frac{1}{(v^2\,t)^m}\;. \tag{65}$$

Thus, the asymptotic inverse of the conventional time migration has the weighting function determined from equations (10) and (62) as

$$w(x;t,y) = \frac{1}{(2\,\pi)^{m/2}}\,\frac{t/z}{v^m\,t^{m/2}}\;. \tag{66}$$

The weighting functions of the asymptotic pseudo-unitary operators are obtained from formulas (28) and (29). They have the form

$$w^{(+)}(x;t,y) \;=\; \frac{1}{(2\,\pi)^{m/2}}\,\frac{\sqrt{t/z}}{v^m\,t^{m/2}}\;. \tag{67}$$

$$w^{(-)}(y;z,x) \;=\; \frac{1}{(2\,\pi)^{m/2}}\,\frac{\sqrt{z/t}}{v^m\,t^{m/2}}\;. \tag{68}$$

The square roots of the cosine factor appearing in formulas (67) and (68) correspond to the analogous terms in the pseudo-unitary Stolt migration proposed by Harlan and Sword (1986).

Figure 1 shows the output of a simple numerical test. The synthetic zero-offset section used in this test is shown in the left plot of Figure 2. The data are taken from Claerbout (1995) and correspond to a synthetic reflectivity model, which contains several dipping layers, a fault, and an unconformity. The input zero-offset section is inverted using an iterative conjugate-gradient method and two different weighting schemes: the uniform weighting and the asymptotic pseudo-unitary weighting (67-68). I compare the iterative convergence by measuring the least-squares norm of the data residual error at different iterations. Figure 1 shows that the pseudo-unitary weighting provides a significantly faster convergence. The result of inversion after 10 conjugate-gradient iterations is shown in Figures 2 and 3. The right plot in Figure 2 shows the output of the least-squares migration. Figure 3 shows the corresponding modeled data and the residual error. The latter is very close to zero. Although this example has only a pedagogical value, it clearly demonstrates possible advantages of using asymptotic pseudo-unitary operators in least-squares migration.

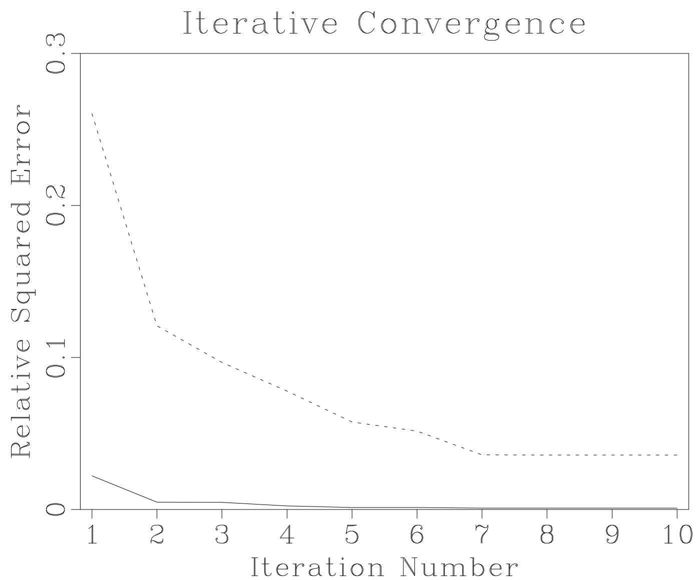

Figure 1: Comparison of convergence of the iterative least-squares migration. The dashed line corresponds to the unweighted (uniformly weighted) operator. The solid line corresponds to the asymptotic pseudo-unitary operator. The latter provides a noticeably faster convergence.

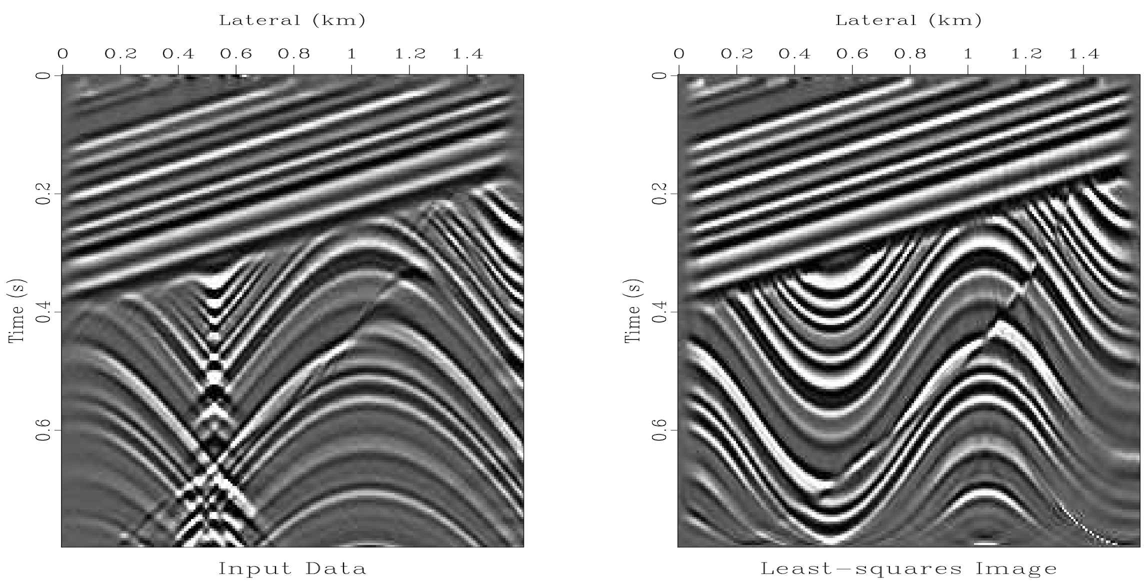

Figure 2: Input zero-offset section (left) and the corresponding least-squares image (right) after 10 iterations of iterative inversion.

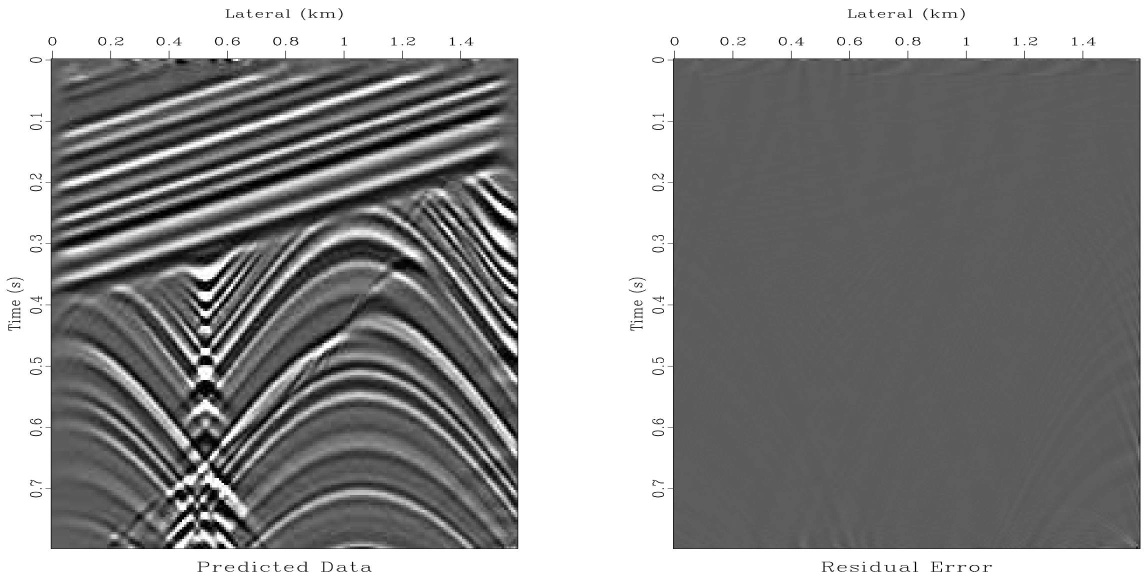

Figure 3: The modeled zero-offset (left) and the residual error (right) plotted at the same scale.

Velocity Transform

Velocity transform is another form of hyperbolic stacking with the summation path

$$\widehat{\theta}(h;t_0,s) = \sqrt{t_0^2 + s^2\,h^2}\;, \tag{69}$$

where h corresponds to the offset, s is the stacking slowness, and t_0 is the estimated zero-offset traveltime. Hyperbolic stacking is routinely applied for scanning velocity analysis in common-midpoint stacking. Velocity transform inversion has proved to be a powerful tool for data interpolation and amplitude-preserving multiple suppression (Thorson, 1984; Ji, 1995; Lumley et al., 1995).

Solving equation (69) for t_0, we find that the asymptotic inverse and adjoint operators have the elliptic summation path

$$\theta(s;t,h) = \sqrt{t^2 - s^2\,h^2}\;. \tag{70}$$

The weighting functions of the asymptotic pseudo-unitary velocity transform are found using formulas (28) and (29) to have the form

$$w^{(+)} = \frac{1}{(2\,\pi)^{1/2}}\,\left|F\,\widehat{F}\right|^{1/4}\,\left|\frac{\partial\widehat{\theta}}{\partial t_0}\right|^{-1/4} = \frac{1}{\sqrt{\pi}}\,\frac{\sqrt{s\,h}\,\sqrt{t/t_0}}{\sqrt{t}}\;. \tag{71}$$

$$w^{(-)} = \frac{1}{(2\,\pi)^{1/2}}\,\left|F\,\widehat{F}\right|^{1/4}\,\left|\frac{\partial\widehat{\theta}}{\partial t_0}\right|^{3/4} = \frac{1}{\sqrt{\pi}}\,\frac{\sqrt{s\,h}\,\sqrt{t_0/t}}{\sqrt{t}}\;. \tag{72}$$

The factor $\sqrt{s\,h}$ for pseudo-unitary velocity transform weighting has been discovered empirically by Claerbout (1995).

Figure 4 shows the output of a numerical test of the least-squares velocity transform inversion using a CMP gather from the Mobil AVO dataset (Lumley et al., 1995). The input CMP gather (shown in the left plot of Figure 5) is inverted using an iterative conjugate-gradient method and two different weighting scheme: the uniform weighting and the asymptotic pseudo-unitary weights (71-72). Analogously to Figure 1, the iterative convergence is measured by the least-squares norm of the data residual error at different iterations. Figure 4 shows that the pseudo-unitary weighting provides a noticeably faster convergence at the first three iterations. At later iterations, the residual errors of the two methods are very close to each other. The use of a pseudo-unitary weighting will be justified in this case if only three iterations are practically affordable. The results of inversion after 10 conjugate-gradient iterations are plotted in Figures 5 and 6. The right plot in Figure 5 shows the output of the velocity transform inversion: an optimized velocity scan. Figure 6 shows the corresponding modeled CMP gather and the residual error. The error is negligible which indicates a successful inversion.

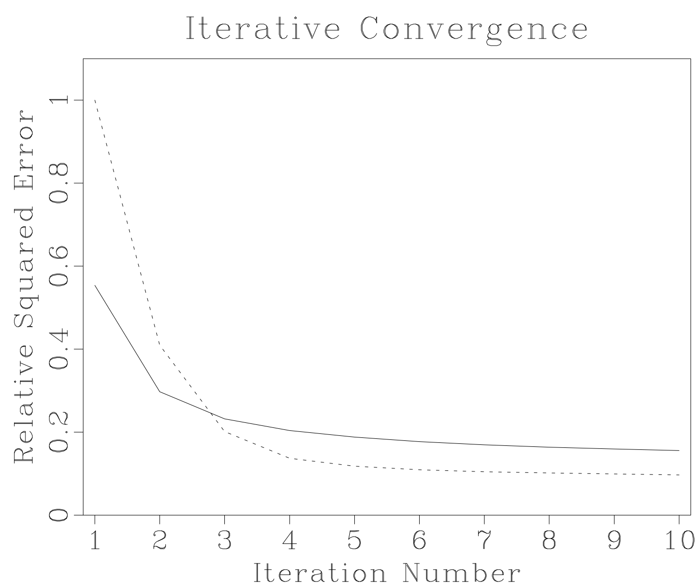

Figure 4: Comparison of convergence of the iterative velocity transform inversion. The dashed line corresponds to the unweighted (uniformly weighted) operator. The solid line corresponds to the asymptotic pseudo-unitary operator. The latter provides a faster convergence at early iterations.

Offset Continuation and DMO

Offset continuation is the operator that transforms seismic reflection data from one offset to another (Bolondi et al., 1982; Salvador and Savelli, 1982). If the data are continued from half-offset h_1 to a larger offset h_2, the summation path of the post-NMO integral offset continuation has the following form (Biondi and Chemingui, 1994; Stovas and Fomel, 1996; Fomel, 2001b):

$$\theta(x;t,y) = \frac{t}{h_2}\sqrt{\frac{U+V}{2}}\ , \tag{73}$$

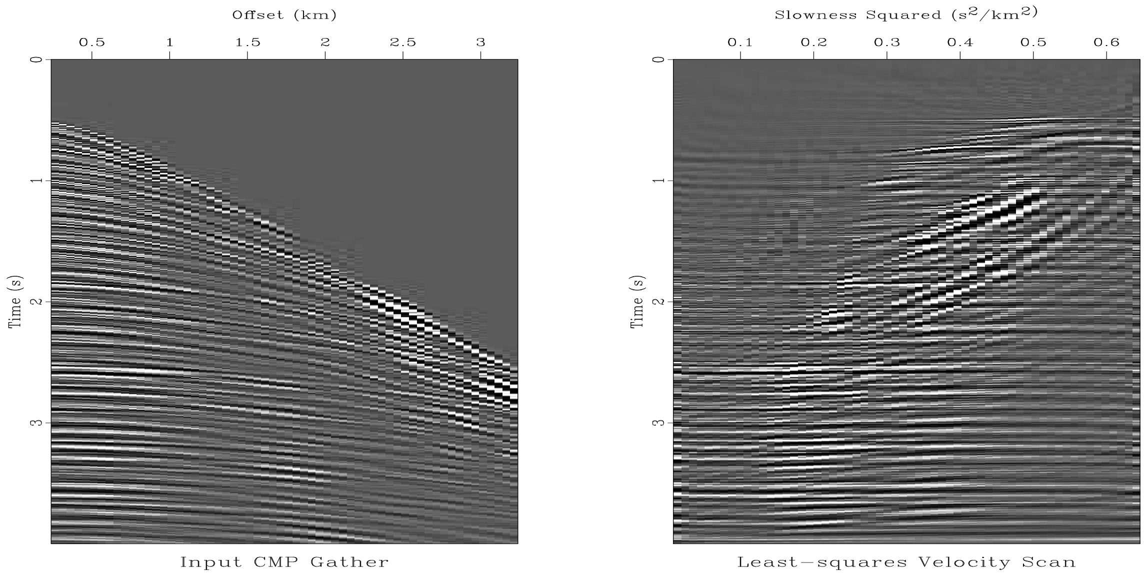

Figure 5: Input CMP gather (left) and its velocity transform counterpart (right) after 10 iterations of iterative least-squares inversion.

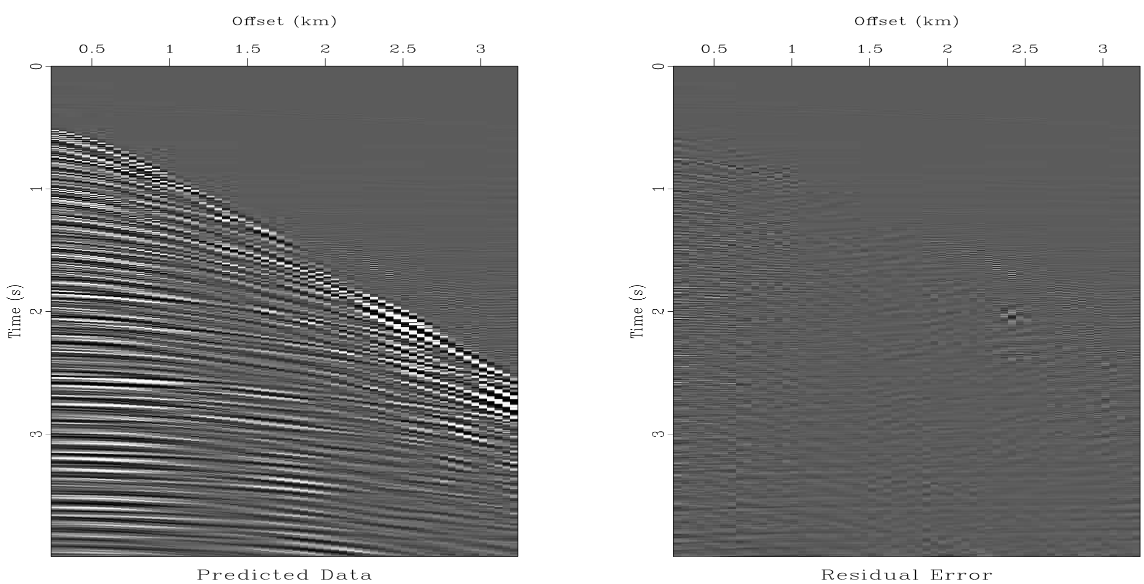

Figure 6: The modeled CMP gather (left) and the residual error (right) plotted at the same scale.

where $U = h_1^2 + h_2^2 - (x-y)^2$, $V = \sqrt{U^2 - 4\,h_1^2\,h_2^2}$, and x and y are the midpoint coordinates before and after the continuation. The summation path of the reverse continuation is found from inverting (73) to be

$$\widehat{\theta}(y;z,x) = z\,h_2\,\sqrt{\frac{2}{U+V}} = \frac{z}{h_1}\,\sqrt{\frac{U-V}{2}}\;. \tag{74}$$

The Jacobian of the time coordinate transformation in this case is simply

$$\left|\frac{\partial\widehat{\theta}}{\partial z}\right| = \frac{t}{z}\;. \tag{75}$$

Differentiating summation paths (73) and (74), we can define the product of the weighting functions according to formula (10), as follows:

$$w\,\widehat{w} = \frac{1}{2\,\pi}\,\sqrt{\left|F\,\widehat{F}\right|\,\left|\frac{\partial\widehat{\theta}}{\partial z}\right|} = \frac{t}{2\,\pi}\,\frac{\left(h_2^2 - h_1^2\right)^2 - (x-y)^4}{V^3}\;. \tag{76}$$

The weighting functions of the amplitude-preserving offset continuation have the form (Fomel, 2001b)

$$w(x;t,y) \;=\; \sqrt{\frac{z}{2\,\pi}}\,\frac{h_2^2 - h_1^2 - (x-y)^2}{V^{3/2}}\;, \tag{77}$$

$$\widehat{w}(y;z,x) \;=\; \frac{t/\sqrt{z}}{\sqrt{2\,\pi}}\,\frac{h_2^2 - h_1^2 + (x-y)^2}{V^{3/2}}\;. \tag{78}$$

It easy to verify that they satisfy relationship (76); therefore, they appear to be asymptotically inverse to each other.

The weighting functions of the asymptotic pseudo-unitary offset continuation are defined from formulas (28) and (29), as follows:

$$w^{(+)} \;=\; \frac{1}{(2\,\pi)^{1/2}}\,\left|F\,\widehat{F}\right|^{1/4}\,\left|\frac{\partial\widehat{\theta}}{\partial t_0}\right|^{-1/4} = \sqrt{\frac{z}{2\,\pi}}\,\frac{\left(\left(h_2^2 - h_1^2\right)^2 - (x-y)^4\right)^{1/2}}{V^{3/2}}\;, \tag{79}$$

$$w^{(-)} \;=\; \frac{1}{(2\,\pi)^{1/2}}\,\left|F\,\widehat{F}\right|^{1/4}\,\left|\frac{\partial\widehat{\theta}}{\partial t_0}\right|^{3/4} = \frac{t/\sqrt{z}}{\sqrt{2\,\pi}}\,\frac{\left(\left(h_2^2 - h_1^2\right)^2 - (x-y)^4\right)^{1/2}}{V^{3/2}}\;. \tag{80}$$

The most important case of offset continuation is the continuation to zero offset. This type of continuation is known as *dip moveout (DMO)*. Setting the initial offset h_1 equal to zero in the general offset continuation formulas, we deduce that the inverse and forward DMO operators have the summation paths

$$\theta(x;t,y) \;=\; \frac{t}{h_2}\,\sqrt{h_2^2 - (x-y)^2}\;, \tag{81}$$

$$\widehat{\theta}(y;z,x) \;=\; \frac{z\,h_2}{\sqrt{h_2^2 - (x-y)^2}}\;. \tag{82}$$

The weighting functions of the amplitude-preserving inverse and forward DMO are

$$w(x;t,y) = \sqrt{\frac{z}{2\,\pi}}\,\frac{1}{h_2}\,, \tag{83}$$

$$\widehat{w}(y;z,x) = \frac{t/\sqrt{z}}{\sqrt{2\,\pi}}\,\frac{h_2\,\left(h_2^2+(x-y)^2\right)}{\left(h_2^2-(x-y)^2\right)^2}\,, \tag{84}$$

and the weighting functions of the asymptotic pseudo-unitary DMO are

$$w^{(+)} = \sqrt{\frac{z}{2\,\pi}}\,\frac{\sqrt{h_2^2+(x-y)^2}}{h_2^2-(x-y)^2}\,, \tag{85}$$

$$w^{(-)} = \frac{t/\sqrt{z}}{\sqrt{2\,\pi}}\,\frac{\sqrt{h_2^2+(x-y)^2}}{h_2^2-(x-y)^2}\,. \tag{86}$$

Equations similar to (83) and (84) have been published by Stovas and Fomel (1996). Equation (84) differs from the similar result of Black et al. (1993) by a simple time multiplication factor. This difference corresponds to the difference in definition of the amplitude preservation criterion. Equation (84) agrees asymptotically with the frequency-domain Born DMO operators (Bleistein, 1990; Liner, 1991; Bleistein and Cohen, 1995). Likewise, the stacking operator with the weighting function (83) corresponds to Ronen's inverse DMO (Ronen, 1987), as discussed by Fomel (2001b). Its adjoint, which has the weighting function

$$\widetilde{w}(x;t,y) = \frac{t/\sqrt{z}}{2\,\pi}\,\frac{1}{h_2}\,, \tag{87}$$

corresponds to Hale's DMO (Hale, 1984).

CONCLUSIONS

Stacking operators such as Kirchoff migration, datuming, dip moveout, velocity transform, etc. are widely used in seismic imaging and data processing, and the need often arises to invert them.

This paper fills the gap between the concept of asymptotically inverse operators and the concept of adjoint operators by introducing the notion of asymptotic pseudo-unitary stacking operators. A pair of asymptotic pseudo-unitary operators possesses the property of being both adjoint and asymptotically inverse to each other. The amplitude (weighting) functions of these operators are completely defined by the derivatives of their kinematics (stacking surfaces).

The practical advantage of this unification is in the ability to construct asymptotically optimal preconditioning for iterative least-squares solution of inverse problems. Simple preliminary tests are encouraging, but further practical experience is needed to confirm the theoretical expectations.

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APPENDIX A

LEAST-SQUARES RADON TRANSFORM INVERSION

This appendix exemplifies the application of adjoint operators by reviewing the analytical least-squares inversion of the classic Radon transform (slant stack operator).

Forming the product $\mathbf{A^T}\,\mathbf{A}$ for this case leads to the double integral

$$\begin{aligned} H(z,x) &= (\mathbf{A^T}\,\mathbf{A})[M(z,x)] = \\ &= \iint \widehat{w}(y;z,x)\, w\left(\xi;\widehat{\theta}(y;z,x),y\right)\, M\left(\theta\left(\xi;\widehat{\theta}(y;z,x),y\right),\xi\right)\,d\xi\,dy = \\ &= \iint M\left(z+y\,(\xi-x)\right)\,d\xi\,dy\;. \end{aligned} \tag{A-1}$$

Applying Fourier transform with respect to z, we can rewrite equation (A-1) in the frequency domain as

$$\check{H}(\omega,x) = \int \check{M}(\omega,\xi) \int e^{i\omega\,y\,(\xi-x)}\,dy\,d\xi\;, \tag{A-2}$$

where

$$\check{H}(\omega,x) = \int H(z,x)\,e^{-i\omega\,z}\,dz\;, \tag{A-3}$$

$$\check{M}(\omega,x) = \int M(z,x)\,e^{-i\omega\,z}\,dz\;. \tag{A-4}$$

The inner integral in equation (A-2) reduces to the m-dimensional delta function:

$$\check{H}(\omega,x) = (2\,\pi)^m \int \check{M}(\omega,\xi)\,\delta\left(\omega^m\,(\xi-x)\right)\,d\xi\;. \tag{A-5}$$

As follows from the properties of delta function,

$$\check{H}(\omega,x) = \frac{(2\,\pi)^m}{|\omega|^m} \int \check{M}(\omega,\xi)\,\delta(\xi-x)\,d\xi\; = \frac{(2\,\pi)^m}{|\omega|^m}\,\check{M}(\omega,x)\;. \tag{A-6}$$

Inverting (A-6) for M, we conclude that

$$\left(\mathbf{A^T\,A}\right)^{\mathbf{-1}} = \frac{|\mathbf{D}|^m}{(2\,\pi)^m}\;. \tag{A-7}$$

Substituting equation (A-7) into (13) produces the result precisely equivalent to Radon's inversion (4).

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